AIAA JOURNAL
Vol. 48, No. 5, May 2010

Technical Notes

Transonic Unsteady Potential Flow: Scaling Analysis of Linear and Nonlinear Dynamics

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DOI: 10.2514/1.40426

I. Introduction

BY ORDER-OF-MAGNITUDE estimates, called scaling analysis [1–4], several interesting questions about unsteady transonic potential flow may be addressed. Similarity laws based upon dimensional analysis have been developed previously for potential-flow models, largely within the context of steady flows [5,6], but also see the interesting discussion of Williams [7,8] on similarity laws in unsteady flows. Such similarity laws are very valuable and tell us how one may express the results of analysis in the most compact dimensionless form. But they do not address the question posed here, which is how large must the airfoil motion be for nonlinear aerodynamic affects to be important, as predicted by potential-flow models? To answer this question, scaling analysis may be used.

Both similarity laws and scaling analysis seek to extract as much information as possible by analysis without resorting to numerical computations. As such, they are useful complements to computations and experiments. As a particular example and a primary motivation for the present work, the limit cycle oscillations of the F-16 aircraft are considered.

Bejan [1] calls the approach employed here "scaling analysis" and he has used this approach very effectively for a wide range of topics in thermodynamics and heat transfer. The present author has independently developed a similar approach for topics in aeroelasticity [2,4].

II. Scaling Analysis

It is well established that for small perturbations, e.g., thin airfoils and/or airfoils undergoing small motions, the governing field equation for the aerodynamic velocity potential in transonic unsteady potential nonlinear flow is as follows: x, y, and z are the conventional Cartesian coordinates in the chordwise, spanwise, and transverse directions,

$$\phi_{xx}[1 - M_L^2] + \phi_{yy} + \phi_{zz} - 1/a_\infty^2[2U_\infty\phi_{xt} + \phi_{tt}] = 0 \tag{1}$$

$$M_L^2 \equiv M_\infty^2\left[1 + (\gamma+1)\frac{\phi_x}{U_\infty}\right] \tag{2}$$

Here, M_∞ is the freestream Mach number and M_L is the local Mach number that varies with x, y, z, and t through its dependence on ϕ_x. Subscripts x, y, and t on ϕ denote derivatives with respect to one of the spatial variables x, y, and z or time t.

To make order-of-magnitude estimates we consider oscillatory flows of frequency ω with a wing of chord c and span ℓ.

The velocity potential ϕ is related to airfoil shape and motion through the small-perturbation boundary condition: that is,

$$\left.\frac{\partial\phi}{\partial z}\right|_{z=0} = w \tag{3}$$

where w is the downwash. Here, the primary interest is with airfoil motion, so for a typical oscillatory angle of attack, the order-of-magnitude estimate of w may be taken as follows:

$$w \sim U_\infty\alpha \tag{4}$$

Equation (4) will be manipulated as though it were an equation, but it is in fact only (but crucially) an order-of-magnitude estimate. Note that there is another term of the order of the reduced frequency that formally appears in Eq. (4) that has been neglected here. Subsequently, it will be shown that reduced frequencies of the order of unity or less are of interest, and hence this term may be omitted when making order-of-magnitude estimates. Also note that we assume that the motion of the airfoil (say, dynamic angle of attack α) is much larger than the thickness ratio of the airfoil, τ, which is true for the results presented here. It would be of interest to consider the case in which α and τ are of the same order of magnitude, but this is left to future work.

III. Linear Aerodynamics

To better fix ideas as a foundation for the discussion of nonlinear dynamics and aerodynamics to follow, first consider the simpler linear theory. For the linear theory, the local Mach number M_L is simply set equal to the freestream Mach number M_∞, which in turn, for our purposes, will be set to 1.

Consider now the order of magnitude of the several terms in Eq. (1). They are as follows:

$$\phi_{xx} \sim \phi/c^2 \tag{5}$$

$$\phi_{yy} \sim \phi/\ell^2 \tag{6}$$

$$\phi_{zz} \sim \phi/\lambda^2 \tag{7}$$

(where λ is a length scale yet to be estimated)

$$\frac{U_\infty}{a_\infty^2}\phi_{xt} \sim \phi\frac{U_\infty}{a_\infty^2}\frac{\omega}{c} \tag{8}$$

Note that in estimating the order of magnitude of the various terms, numerical factors of order 1, such as $(\gamma + 1)$, may be ignored:

$$\frac{1}{a_\infty^2}\phi_{tt} \sim \phi\frac{\omega^2}{a_\infty^2} \tag{9}$$

By taking the ratio of Eqs. (6–9) to Eq. (5) one may estimate the relative size of the several terms. The results are as follows:

Ratio of Eq. (6) to Eq. (5):

$$\sim (c/\ell)^2 \tag{10}$$

Ratio of Eq. (7) to Eq. ,(5):

$$\sim (c/\lambda)^2 \tag{11}$$

Received 26 August 2008; revision received 1 October 2009; accepted for publication 31 January 2010.

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Ratio of Eq. (8) to Eq. (5):

$$\sim M_\infty\left(\frac{\omega c}{a_\infty}\right) \tag{12}$$

Ratio of Eq. (9) to Eq. (5):

$$\sim\left(\frac{\omega c}{a_\infty}\right)^2 \tag{13}$$

Again recall that for our purposes, M_∞ is taken to be of order 1 and set to 1 since our interest is in transonic flow, but the cases $M_\infty \ll 1$ or $M_\infty \gg 1$ could be considered by similar methods. In the general case, Eqs. (10–13) are all of unit order of magnitude. When one or more of these ratios are substantially less than unity, certain simplifications may be made, as will be discussed next.

So what may one conclude from Eqs. (10–13)? A number of well-known and perhaps not-so-well-known results may be obtained.

First, if

$$\frac{\omega c}{a_\infty} \sim 1 \tag{14}$$

then the flow is significantly unsteady and the terms involving time derivatives (9) must be retained, which is a reassuring but hardly surprising result. If $\omega c/a_\infty \ll 1$, then unsteadiness may be neglected or at least is of modest importance.

Second, if

$$c/\ell \ll 1 \tag{15}$$

then the flow is two-dimensional and the ϕ_{yy} term (6) may be neglected, which is a well-known result.

Third, if

$$c/\ell \gg 1 \tag{16}$$

then the ϕ_{xx} term may be neglected and if

$$\frac{\omega \ell}{a_\infty} \sim 1 \tag{17}$$

then

$$\left(\frac{\omega c}{a_\infty}\right) \sim \left(\frac{\omega \ell}{a_\infty}\right)\left(\frac{c}{\ell}\right) \ll 1 \tag{18}$$

and the time derivative terms (8) and (9) may be neglected, which reduces Eq. (1) to

$$\phi_{yy} + \phi_{zz} = 0 \tag{19}$$

This is the well-known slender-body, or slender-wing, theory seen from a new perspective based upon scaling analysis.

Finally, consider

$$\frac{\omega c}{a_\infty} \gg 1$$

Then Eq. (1) reduces to the following:

$$\phi_{zz} - \frac{1}{a_\infty^2}\phi_{tt} = 0$$

which is the well-known linear piston theory result.

Now what about Eq. (11)? This allows one to estimate λ, the scale of the flow in the z direction. To see this most clearly, return to Eqs. (5–9) and now construct the ratios of Eqs. (5), (6), (8), and (9) to Eq. (7), as follows.

Ratio of Eq. (5) to Eq. (7):

$$\sim (\lambda/c)^2 \tag{20}$$

Ratio of Eq. (6) to Eq. (7):

$$\sim (\lambda/\ell)^2 \tag{21}$$

Ratio of Eq. (8) to Eq. (7):

$$\sim M_\infty\left(\frac{\lambda\omega}{a_\infty}\right)\lambda/c \tag{22}$$

Ratio of Eq. (9) to Eq. (7):

$$\sim\left(\frac{\lambda\omega}{a_\infty}\right)^2 \tag{23}$$

The largest of these ratios determines λ, for the ϕ_{zz} term must balance the largest of the other terms. Thus, for example, assuming $\omega c/a_\infty \sim 1$, then if

$$(c/\ell) \gg 1 \tag{24}$$

it follows that $\lambda \sim \ell$, and if

$$(c/\ell) \ll 1 \tag{25}$$

it follows that $\lambda \sim c$.

On the other hand, if

$$(c/\ell) \sim 1 \quad \text{and} \quad \left(\frac{\omega c}{a_\infty}\right) \gg 1 \tag{26}$$

then it follows that

$$\lambda \sim a_\infty/\omega$$

IV. Nonlinear Dynamics and Aerodynamics

It is the estimate of λ that is a key to considering the nonlinear case, with M_L, the local Mach number, now given by Eq. (2).

Consider now the nonlinear term in Eq. (1) and again set the freestream Mach number, $M_\infty = 1$, for our purposes. An order-of-magnitude estimate for this nonlinear term is as follows: Note again that $\gamma + 1$ is of order 1 for the purposes of scaling analysis:

$$\phi_{xx}\frac{\phi x}{U_\infty} \sim \phi_{xx}\frac{\phi_x}{a_\infty} \sim \frac{\phi^2}{a_\infty c^3} \tag{27}$$

Now compare this term to ϕ_{zz}; recall from Eq. (6) that $\phi_{zz} \sim \phi/\lambda^2$. If Eq. (27) is of the same order as Eq. (6) or greater, the nonlinear term must be retained; conversely, if it is much smaller, it may be safely neglected. Thus, construct the ratio of Eq. (27) to Eq. (6):

$$\frac{\phi^2}{a_\infty c^3} \Big/ \phi/\lambda^2 \sim \frac{\phi\lambda^2}{a_\infty c^3} \tag{28}$$

Thus, the critical order of magnitude of ϕ is determined as follows:

$$\phi_{\mathrm{cr}} \sim a_\infty\, c^3/\lambda^2 \tag{29}$$

For $\phi < \phi_{\mathrm{cr}}$, linear theory is sufficient, and if $\phi \gg \phi_{\mathrm{cr}}$, a nonlinear theory is required. But how do we determine λ?

Again, we require ϕ_{zz} to balance the other linear terms. In particular, we ask that it balance the remaining *dominant* linear term. So consider each remaining linear term in turn:

$$\phi_{zz} \sim \phi_{yy} \Rightarrow \frac{\phi}{\lambda^2} \sim \phi/\ell^2 \Rightarrow \lambda = \ell \tag{30}$$

or

$$\phi_{zz} \sim \frac{U_\infty}{a_\infty^2}\phi_{xt} \Rightarrow \frac{\phi}{\lambda^2} \sim \frac{\phi\omega}{a_\infty c} \Rightarrow \lambda = \sqrt{\frac{a_\infty c}{\omega}} \tag{31}$$

or

$$\phi_{zz} \sim \frac{1_\infty}{a_\infty^2}\phi_{tt} \Rightarrow \frac{\phi}{\lambda^2} \sim \frac{\phi\omega^2}{a_\infty^2} \Rightarrow \lambda = \frac{a_\infty}{\omega} \tag{32}$$

Now Eq. (30) will be appropriate to the slender-body limit (31) to the two-dimensional flow limit at low frequencies, and Eq. (32) will be appropriate to two-dimensional flows at high frequencies.

Using Eqs. (30–32) in turn in Eq. (29), one can deduce the following.

From Eqs. (29) and (30),

$$\phi_{\mathrm{cr}} \sim a_\infty c^3/\ell^2 \sim a_\infty c(c/\ell) \tag{33}$$

which tells us that ϕ_{cr} is far beyond the range of small-perturbation theory itself for small-aspect-ratio wings $\phi \ll \alpha_\infty c$, or slender-wing theory. Recall that for small-perturbation theory to be applicable, one requires that $c/\ell \geq 1$. Hence, the slender-body limit of small-perturbation potential flow cannot model aerodynamic nonlinearities accurately.

From Eqs. (29) and (31),

$$\phi_{\mathrm{cr}} \sim a_\infty c \frac{c^2}{a_\infty c/\omega} \sim a_\infty c \left(\frac{\omega c}{a_\infty}\right) \tag{34}$$

so that only for small $\omega c/a_\infty$ will ϕ_{cr} be much less than $a_\infty c$ and within the range of small-perturbation theory, i.e., $\phi_{\mathrm{cr}} \ll \alpha_\infty c$.

From Eqs. (29) and (32),

$$\phi_{\mathrm{cr}} \sim a_\infty c^3/\lambda^2 \sim a_\infty c^3 \frac{\omega^2}{a_\infty^2}$$

or

$$\phi_{\mathrm{cr}} \sim \alpha_\infty c \left(\frac{\omega c}{\alpha_\infty}\right)^2 \tag{35}$$

and thus for such a ϕ_{cr}, small-perturbation theory will again be violated, since Eq. (32) assumes $\omega c/a_\infty \gg 1$, and hence from Eq. (35),

$$\phi_{\mathrm{cr}} \gg \alpha_\infty c$$

Thus, of the three cases, Eqs. (30–32), only the case of Eq. (31) (i.e., $\omega c/a_\infty$ of order one or less) is of remaining interest, as anticipated in the earlier discussion.

V. Estimate of Critical Magnitude of Motion for Potential-Flow Nonlinearities to be Important

For the case of Eq. (31), what is the critical size of the motion or angle of attack α_{cr} corresponding to ϕ_{cr}?

From Eqs. (3) and (4),

$$\left.\frac{\partial \phi}{\partial z}\right|_{z=0} = w, \qquad w \sim U_\infty \alpha$$

or

$$\frac{\phi}{\lambda} \sim U_\infty \alpha \tag{36}$$

From Eqs. (31), (34), and (36),

$$\alpha_{\mathrm{cr}} = \frac{\phi_{\mathrm{cr}}}{\lambda U_\infty} = \frac{a_\infty c}{U_\infty} \frac{(\omega c/a_\infty)}{\sqrt{a_\infty c/\omega}} = \left(\frac{\omega c}{a_\infty}\right)^{3/2} \tag{37}$$

But recall that $\omega c/a_\infty \leq 1$ for Eqs. (31), (34), and (37) to be valid and that $M_\infty = U_\infty/a_\infty$ is again set to 1. Note that if $\omega = 0$, then nonlinearities are important for any angle of attack, i.e., $\alpha_{\mathrm{cr}} = 0$. This is also a well-known result from classical steady flow transonic aerodynamics [5,6].

Consider now a numerical example representative of a high-performance aircraft such as the F-16. Choose the following parameter values: $\omega/2\pi \sim 10$ Hz, $c \sim 10$ ft, and $U_\infty = a_\infty \sim 1000$ ft/s. Thus, $\omega c/a_\infty \sim 0.6$. From Eq. (37),

$$\alpha_{\mathrm{cr}} \sim \left(\frac{\omega c}{a_\infty}\right)^{3/2} \sim 0.5 \text{ rad} \sim 30^\circ$$

This angle of attack for pitch oscillations is much larger than the motions described as limit cycle oscillations (LCOs) that are found in flight tests for fighter aircraft and also in computational models [9]. Hence, this result is consistent with the conclusions drawn by Dowell et al. [9] that an inviscid potential-flow model without some modeling of the viscous boundary layer is not capable of describing LCOs for current aircraft of interest. A Navier–Stokes model that includes the effects of separated flow predicts LCO amplitudes that are typically an order of magnitude smaller than the α_{cr} determined by an inviscid potential-flow model, for example [9].

Note, by the way, that the thickness ratio τ of the F-16 is of the order of 0.05, i.e., an order of magnitude less than α_{cr} for this numerical example, consistent with the assumptions of the present scaling analysis.

VI. Conclusions

The scaling analysis reported here supports the conclusions reached by numerical computations and flight experiments regarding the inability of inviscid potential-flow models to predict LCO motions per se. Of course, inviscid potential-flow models may still be useful in predicting the conditions for the onset of LCO and flutter as Landahl suggested many years ago in his classic book [10]. Finally, note that although the governing equations of transonic small-disturbance potential-flow theory are usually derived by other methods, they may be deduced from full potential-flow theory by scaling analysis [11].

An interesting and open question remains, however. Can one use scaling analysis to predict the order of magnitude of observed LCO motions based upon the Euler or Navier–Stokes equations?

Finally, the author notes that the essence of scaling analysis has been developed and used by the author to develop order-of-magnitude estimates of LCO aeroelastic response for flat and curved plates and also for high-altitude long-endurance (HALE) wings [2–4]. Bejan [1], the author's colleague, has independently developed similar ideas and used them in the scaling analysis of thermodynamic systems.

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